

Theory of Exponential Splines*

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Pruess [12, 14] has shown that exponential splines can produce co-convex and co-monotone interpolants. These results justify the further study of the mathematical properties of exponential splines as they pertain to their utility as numerical approximations. They also warrant the generalization of the exponential spline in fruitful directions. Herein, we present convergence rates and extremal properties for exponential spline approximation, cardinal spline and B-spline bases for the space of exponential splines, and generalizations to higher order tension splines and Hermite tension interpolants. © 1991 Academic Press, Inc.

1. INTRODUCTION

In this paper, we discuss exponential splines from a theoretical viewpoint. The importance of such an investigation is underscored by Pruess' results [12, 14] asserting that exponential splines can produce co-convex and co-monotone interpolants. The utility of such approximants for the applications is quite clear.

Starting from the analogy of a cubic spline to a beam, we add a tension term to the governing differential equation thus giving rise to the exponential spline. The solution to this boundary value problem expresses the exponential spline in terms of its second derivative at the knots. However, there is an alternative representation of the exponential spline in terms of its first derivative at the knots. For future reference as well as completeness, we next derive this other system of equations. Pruess' results on the shape preservation capabilities of exponential splines are then reviewed since they provide the *raison d'être* for what follows.

Convergence of the approximating spline is next studied through the proximity of the interpolating cubic and exponential splines with identical

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end conditions. We then introduce two bases for the space of exponential splines, namely the cardinal spline basis and the B-spline basis. A discussion of the appropriateness of each of these representations is included.

The exponential spline is next inspected in the context of generalized splines. As a consequence of this perspective, we obtain certain extremal properties for exponential splines in two naturally arising pseudonorms. We show how to combine these results to produce extremal properties in a third pseudonorm. Finally, we are led to consider higher order tension splines and higher degree interpolation by piecewise exponentials.

2. NOTATION

For ease of reference, we here collect the principal notation used in this paper:

$$a = x_1 < x_2 < \cdots < x_N < x_{N+1} = b, \quad h_i = x_{i+1} - x_i \quad (i = 1, \dots, N)$$

$$h = \max_{1 \leq i \leq N} h_i, \quad p = \max_{1 \leq i \leq N} p_i, \quad s = \sinh(ph), \quad c = \cosh(ph)$$

$$b_1 = \frac{f_2 - f_1}{h_1} - f'(a), \quad b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} \quad (i = 2, \dots, N),$$

$$b_{N+1} = f'(b) - \frac{f_{N+1} - f_N}{h_N}$$

$$m_0 = f'(a), \quad m_i = \frac{f_{i+1} - f_i}{h_i} \quad (i = 1, \dots, N), \quad m_{N+1} = f'(b)$$

$$s_i = \sinh(p_i h_i), \quad c_i = \cosh(p_i h_i) \quad (i = 1, \dots, N)$$

$$d_i = \left[p_i \frac{c_i}{s_i} - \frac{1}{h_i} \right] / p_i^2, \quad e_i = \left[\frac{1}{h_i} - \frac{p_i}{s_i} \right] / p_i^2 \quad (i = 1, \dots, N).$$

3. EXPONENTIAL SPLINE EQUATIONS

The cubic spline is well known to have the following analogue in beam theory [1]. Consider a simply supported beam with supports $\{(x_i, f_i)\}_{i=1}^{N+1}$. Then $s(x)$, the deflection of the beam, is a solution to the differential equation $[E \cdot I \cdot D^2]s = M$ between successive supports. Here E = Young's modulus, I = cross-sectional moment of inertia, M = bending moment. Under the assumption of weightlessness, M is a piecewise-linear continuous function with break points at the supports. Differentiating the

above twice, we arrive at the two-point boundary value problem on $[x_i, x_{i-1}]$ ($i = 1, \dots, N$)

$$[D^4]s = 0, \quad s(x_i) = f_i, s(x_{i-1}) = f_{i+1}, s''(x_i) = s_i'', s''(x_{i-1}) = s_{i+1}'', \quad (3.1)$$

where s_i'' and s_{i+1}'' are chosen to ensure $s \in C^2[a, b]$ when $s'(a)$ and $s'(b)$ are given. Note that $[D^4]s = 0$ on $[x_i, x_{i-1}]$ implies that s is a cubic there.

The cubic spline so defined has a tendency to exhibit unwanted undulations. The above analogy suggests that the application of uniform tension between supports might remedy the problem [18]. The beam equation on $[x_i, x_{i+1}]$ then becomes $[E \cdot I \cdot D^2 - t_i I]s = M$. In the following paragraphs, $s(x)$ will denote the cubic spline and $\tau(x)$ the exponential spline.

Letting $p_i^2 = t_i E \cdot I$, the above considerations lead us to define the exponential spline [19, 20] as the solution to the boundary value problem on $[x_i, x_{i-1}]$ ($i = 1, \dots, N$)

$$[D^4 - p_i^2 D^2]\tau = 0, \quad \tau(x_i) = f_i, \tau(x_{i+1}) = f_{i+1}, \\ \tau''(x_i) = \tau_i'', \tau''(x_{i+1}) = \tau_{i+1}'', \quad (3.2)$$

with τ_i'' ($i = 1, \dots, N+1$) as yet undetermined. Let us pause to note that $p_i \rightarrow 0 \Rightarrow [D^4]\tau = 0$ (cubic spline) while $p_i \rightarrow \infty \Rightarrow [D^2]\tau = 0$ (linear spline).

The solution to this boundary value problem is

$$\tau(x) = \frac{1}{p_i^2 s_i} \{ \tau_i'' \sinh p_i(x_{i+1} - x) + \tau_{i+1}'' \sinh p_i(x - x_i) \} \\ + \left[f_i - \frac{\tau_i''}{p_i^2} \right] \cdot \frac{x_{i+1} - x}{h_i} + \left[f_{i+1} - \frac{\tau_{i+1}''}{p_i^2} \right] \cdot \frac{x - x_i}{h_i} \quad (3.3)$$

$$= A_i + B_i x + C_i e^{p_i x} + D_i e^{-p_i x}. \quad (3.4)$$

The requirement of first derivative continuity at the points of interpolation yields expressions for the determination of τ_i'' ($i = 1, \dots, N+1$). Specifically, τ_i'' ($i = 1, \dots, N+1$) are the solution to the system of equations

$$d_1 \tau_1'' + e_1 \tau_2'' = b_1 \\ e_{i-1} \tau_{i-1}'' + (d_{i-1} + d_i) \tau_i'' + e_i \tau_{i+1}'' = b_i \quad (i = 2, \dots, N) \quad (3.5) \\ e_N \tau_N'' + d_N \tau_{N+1}'' = b_{N+1},$$

where the first and the last equations represent specified slope end conditions, $\tau'(a) = f'(a)$, $\tau'(b) = f'(b)$. If either τ_1'' or τ_{N+1}'' is specified, it is simply eliminated from (3.5). $\tau(x)$ is uniquely defined once τ_i'' ($i = 1, \dots, N+1$) are determined.

As an alternative to the above formulation of the exponential spline in terms of second derivatives, we have the following formulation in terms of first derivatives. In this case, on $[x_i, x_{i+1}]$ the exponential spline is the solution to the boundary value problem

$$[D^4 - p_i^2 D^2]\tau = 0, \quad \tau(x_i) = f_i, \tau(x_{i+1}) = f_{i+1}, \tau'(x_i) = \tau'_i, \tau'(x_{i+1}) = \tau'_{i+1}. \quad (3.6)$$

The general solution of (3.6) is

$$\tau(x) = a + bx + ce^{p_i(x_{i+1}-x)} + de^{p_i(x-x_i)}. \quad (3.7)$$

Differentiating (3.7), we obtain

$$\tau'(x) = b - p_i ce^{p_i(x_{i+1}-x)} + p_i de^{p_i(x-x_i)}. \quad (3.8)$$

The determination of a , b , c , d then consists of enforcing the above boundary conditions, (3.7).

This results in

$$\begin{aligned} \tau(x) = & f_i \cdot \frac{x_{i+1}-x}{h_i} + f_{i+1} \cdot \frac{x-x_i}{h_i} + \frac{1}{e_i-d_i} \cdot \frac{f_{i+1}-f_i}{h_i} \\ & \cdot \left[\frac{\sinh p_i(x-x_i) - \sinh p_i(x_{i+1}-x)}{p_i^2 s_i} + \frac{x_{i+1}-2x+x_i}{p_i^2 h_i} \right] \\ & + \tau'_i \left\{ \frac{d_i}{e_i^2-d_i^2} \cdot \left[\frac{\sinh p_i(x_{i+1}-x)}{p_i^2 s_i} - \frac{x_{i+1}-x}{p_i^2 h_i} \right] \right. \\ & \left. - \frac{e_i}{e_i^2-d_i^2} \cdot \left[\frac{\sinh p_i(x-x_i)}{p_i^2 s_i} - \frac{x-x_i}{p_i^2 h_i} \right] \right\} \\ & + \tau'_{i+1} \left\{ \frac{e_i}{e_i^2-d_i^2} \cdot \left[\frac{\sinh p_i(x_{i+1}-x)}{p_i^2 s_i} - \frac{x_{i+1}-x}{p_i^2 h_i} \right] \right. \\ & \left. - \frac{d_i}{e_i^2-d_i^2} \cdot \left[\frac{\sinh p_i(x-x_i)}{p_i^2 s_i} - \frac{x-x_i}{p_i^2 h_i} \right] \right\} \\ & (i = 1, \dots, N). \end{aligned} \quad (3.9)$$

Specifying $\tau_1'' = f''(a)$ and $\tau_{N+1}'' = f''(b)$, τ'_i ($i = 1, \dots, N+1$) are the solution to the tridiagonal system

$$\begin{aligned}
& \left[\frac{d_1}{d_1^2 - e_1^2} \right] \tau'_1 + \left[\frac{e_1}{d_1^2 - e_1^2} \right] \tau'_2 = \left[\frac{1}{d_1 - e_1} \right] \left[\frac{f_2 - f_1}{h_1} \right] - \tau''_1 \\
& \left[\frac{e_{i-1}}{d_{i-1}^2 - e_{i-1}^2} \right] \tau'_{i-1} + \left[\frac{d_{i-1}}{d_{i-1}^2 - e_{i-1}^2} + \frac{d_i}{d_i^2 - e_i^2} \right] \tau'_i + \left[\frac{e_i}{d_i^2 - e_i^2} \right] \tau'_{i+1} \\
& = \left[\frac{1}{d_{i-1} - e_{i-1}} \right] \left[\frac{f_i - f_{i-1}}{h_{i-1}} \right] + \left[\frac{1}{d_i - e_i} \right] \left[\frac{f_{i-1} - f_i}{h_i} \right] \quad (i = 2, \dots, N) \\
& \left[\frac{e_N}{d_N^2 - e_N^2} \right] \tau'_N + \left[\frac{d_N}{d_N^2 - e_N^2} \right] \tau'_{N+1} = \tau''_{N+1} + \left[\frac{1}{d_N - e_N} \right] \left[\frac{f_{N+1} - f_N}{h_N} \right]
\end{aligned} \tag{3.10}$$

which is an expression of the C^2 -smoothness requirement. If either τ'_1 or τ'_{N+1} is specified, it is simply eliminated from (3.10).

4. SHAPE PRESERVATION AND CONVERGENCE

We now review the results of Pruess [12, 14] concerning the behavior of exponential splines in the limit of infinite tension. Of primary interest are the shape preservation properties of exponential spline interpolation. We next establish rates of convergence for the exponential spline in the limit of vanishing mesh width. Convergence rates for higher derivatives are also given for functions possessing an appropriate degree of smoothness.

Let $\lambda(x)$ denote the linear spline of interpolation. Then we have

THEOREM 4.1. *Given a sequence of exponential splines such that $p_i \rightarrow \infty$ for some i , then $\tau''(x) \rightarrow 0$ and $\tau'(x) \rightarrow \lambda'(x)$ uniformly in any closed subinterval of (x_i, x_{i+1}) , while $\tau(x) \rightarrow \lambda(x)$ uniformly in $[x_i, x_{i+1}]$.*

This theorem gives us hope that we can produce co-convex and co-monotone interpolants using exponential splines with sufficiently high tension. The fulfillment of this expectation is the subject of

THEOREM 4.2. *If b_i, b_{i+1} are positive (negative), then, for p_{i-1}, p_i, p_{i+1} sufficiently large, $\tau''(x)$ is positive (negative) in $[x_i, x_{i+1}]$. If $\lambda'(x)$ is positive (negative) in (x_{i-1}, x_{i+2}) , then, for p_{i-1}, p_i, p_{i+1} sufficiently large, $\tau'(x)$ is positive (negative) in $[x_i, x_{i+1}]$.*

The cubic spline many times exhibits unwanted oscillations in the form of overshoots and/or extraneous inflection points. The above results assure us that the exponential spline can remedy this situation for appropriately chosen tension parameters. See McCartin [8] for details.

We now establish rates of convergence for the exponential spline in the limit of vanishing mesh width. Convergence rates for higher derivatives are also provided for functions possessing the required degree of smoothness.

We begin by studying the proximity of cubic and exponential splines with identical end conditions. The following result has been established by Pruess [12].

THEOREM 4.3.

$$\|D^i(s - \tau)\|_\infty \leq (26/3) p^2 h^{4-i} \max_j |s_j''| \quad (i=0, 1, 2). \quad (4.1)$$

Here, as before, s and τ are the cubic and exponential splines, respectively. We next present a companion result.

THEOREM 4.4.

$$\|D^i(s - \tau)\|_\infty \leq (26/3) p^2 h^{4-i} \max_j |\tau_j''| \quad (i=0, 1, 2). \quad (4.2)$$

Proof. Similar to that of Theorem 4.3 (see [7]). ■

These theorems may be used to obtain results for $\|D^i(f - \tau)\|_\infty$ from known bounds for $\|D^i(f - s)\|_\infty$. For example,

COROLLARY 4.1. *If $f(x) \in C^4[a, b]$ then there exists a constant, k , independent of h such that $\|D^i(f - \tau)\| \leq kh^{4-i}$ ($i=0, 1, 2$).*

We now take up the convergence of the third derivative. Defining $\delta = s - \tau$ and $\Delta = [\delta_1'', \dots, \delta_{N+1}'']^T$, we have on $[x_i, x_{i+1}]$

$$\begin{aligned} \delta'''(x) = & \frac{\delta_{i+1}'' - \delta_i''}{h_i} + \tau_{i+1}'' \left[\frac{1}{h_i} - \frac{p_i}{s_i} \cosh p_i(x - x_i) \right] \\ & + \tau_i'' \left[\frac{-1}{h_i} + \frac{p_i}{s_i} \cosh p_i(x_{i+1} - x) \right], \end{aligned} \quad (4.3)$$

where we have used the identity that $s'''(x) = (s_{i+1}'' - s_i'')/h_i$ on $[x_i, x_{i+1}]$. The fact that the bracketed expressions in (4.3) are monotonic on $[x_i, x_{i+1}]$, combined with the inequality

$$0 \leq \frac{1}{h_i} - \frac{p_i}{s_i} \leq \frac{1}{2} \left[\frac{p_i c_i}{s_i} - \frac{1}{h_i} \right] \Rightarrow \quad (4.4)$$

$$|\delta'''(x)| \leq \frac{2}{h_i} \|\Delta\|_\infty + \left[\frac{p_i c_i}{s_i} - \frac{1}{h_i} \right] \cdot \frac{3}{2} \max_i |\tau_i''|. \quad (4.5)$$

However,

$$\frac{p_i c_i}{s_i} - \frac{1}{h_i} \leq \frac{p^2 h_i}{3}. \quad (4.6)$$

Therefore, (4.5) yields

$$\begin{aligned} |\delta'''(x)| &\leq \frac{52}{3} p^2 \frac{h^2}{h_i} \max_i |\tau_i''| + \frac{1}{2} p^2 h_i \max_i |\tau_i''| \\ &\leq \frac{107}{6} p^2 h \max_i |\tau_i''| \cdot \frac{h}{h_{\min}}, \end{aligned} \quad (4.7)$$

where $h_{\min} = \min_i h_i$.

Known results for the cubic spline (see, e.g., [4, Theorem 2]) now establish the $O(h)$ convergence of the third derivative. Specifically,

THEOREM 4.5. *If $f(x) \in C^4[a, b]$ then there exists a constant, k , independent of h but dependent on h/h_{\min} such that $\|D^3(f - \tau)\| \leq kh$.*

Proof.

$$\begin{aligned} \|D^3(f - \tau)\|_{\infty} &\leq \|D^3(f - s)\|_{\infty} + \|D^3(s - \tau)\|_{\infty} \\ &\leq \|D^4 f\|_{\infty} \cdot k_1 \cdot h + \frac{107}{6} p^2 \cdot \frac{h}{h_{\min}} \cdot \max_i |\tau_i''| \cdot h = kh. \end{aligned} \quad (4.8)$$

where k_1 and hence k depend on h/h_{\min} . ■

COROLLARY 4.2. *If p and h/h_{\min} are bounded as $h \rightarrow 0$ then $\|D^3(f - \tau)\|_{\infty} = O(h)$.*

5. EXPONENTIAL SPLINE BASES

Both theoretical and practical aspects of exponential splines are greatly illuminated by representation in terms of simple basis splines. In the following paragraphs, two of the most useful bases, the cardinal splines and the B-splines, are introduced and studied for the case of uniform mesh and tension.

The cardinal spline basis $\{C_i(x)\}_{i=0}^{N+2}$ is uniquely defined by the following conditions [3, 4, 11, 15].

$$\left. \begin{aligned} C'_0(x_1) &= 1, & C_0(x_j) &= 0, & C'_0(x_{N+1}) &= 0 \\ C'_i(x_1) &= 0, & C_i(x_j) &= \delta_{ij}, & C'_i(x_{N+1}) &= 0 \\ C'_{N-2}(x_1) &= 0, & C_{N+2}(x_j) &= 0, & C'_{N+2}(x_{N+1}) &= 1 \end{aligned} \right\} \quad (i, j = 1, \dots, N+1). \quad (5.1)$$

Clearly,

$$\tau(x) = f'_1 \cdot C_0(x) + \sum_{i=1}^{N+1} f_i \cdot C_i(x) + f'_{N+1} \cdot C_{N+2}(x). \quad (5.2)$$

The principal structural properties of the cardinal basis splines are next derived. We begin by generalizing the arguments of Birkhoff and de Boor [4]. Let $t(x)$ be the function on $[x_1, x_{N+1}]$ of the form

$$t(x) = a + bx + ce^{px} + de^{-px} \quad (5.3)$$

satisfying

$$\begin{aligned} t(x_1) &= f(x_1), \quad t(x_{N+1}) = f(x_{N+1}), \\ t'(x_1) &= f'(x_1), \quad t'(x_{N+1}) = f'(x_{N+1}). \end{aligned} \quad (5.4)$$

Given the exponential spline fit to $g(x) = f(x) - t(x)$ with zero slope end conditions, we can simply add $t(x)$ to it to obtain the fit for $f(x)$. Hence, without loss of generality, we consider only functions $f(x)$ such that $f(x_1) = f(x_{N+1}) = f'(x_1) = f'(x_{N+1}) = 0$. Thus, we need only consider

$$\tau(x) = \sum_{i=2}^N f_i \cdot C_i(x). \quad (5.5)$$

The following results can be established by arguments nearly identical to those in [4]. Consequently, their proofs are either omitted or abbreviated. Full details are available in [7].

LEMMA 5.1. *Any function of the form $e(x) = a + bx + ce^{px} + de^{-px}$ which satisfies $e(0) = 0$ and $e(h) = 0$ also satisfies*

$$\begin{bmatrix} e'(h) \\ e''(h) \end{bmatrix} = \begin{bmatrix} \frac{phc-s}{ph-s} & \frac{2(1-c)+phs}{p(ph-s)} \\ \frac{p^2hs}{ph-s} & \frac{phc-s}{(ph-s)} \end{bmatrix} \begin{bmatrix} e'(0) \\ e''(0) \end{bmatrix}. \quad (5.6)$$

Proof.

$$\begin{aligned} e(x) &= -\frac{e''_0}{p^2} + x \cdot \left[\frac{e''_0(1-c) - e'_0(ps)}{p(ph-s)} \right] + \sinh px \\ &\quad \cdot \left[\frac{e''_0(c-1) + e'_0(p^2h)}{p^2(ph-s)} \right] + \cosh px \cdot \left[\frac{e''_0}{p^2} \right], \end{aligned} \quad (5.7)$$

where $e'_0 = e'(0)$ and $e''_0 = e''(0)$. ■

COROLLARY 5.1. For $i \neq j + 1, j$, $C_i(x)$ satisfies

$$\begin{bmatrix} C'_i(x_{j+1}) \\ C''_i(x_{j+1}) \end{bmatrix} = \begin{bmatrix} \frac{phc-s}{ph-s} & \frac{2(1-c)+phs}{p(ph-s)} \\ \frac{p^2hs}{ph-s} & \frac{phc-s}{(ph-s)} \end{bmatrix} \begin{bmatrix} C'_i(x_j) \\ C''_i(x_j) \end{bmatrix}, \quad (5.8)$$

COROLLARY 5.2. For $i = 2, \dots, N$, $C_i(x)$ satisfies

$$\begin{aligned} C'_i(x_j) C''_i(x_j) &\geq 0, & j < i \\ C'_i(x_j) C''_i(x_j) &\leq 0, & j > i. \end{aligned} \quad (5.9)$$

COROLLARY 5.3. For $i = 2, \dots, N$, $C_i(x)$ satisfies

$$\begin{aligned} |C'_i(x_j)| &< \frac{1}{2} |C'_i(x_{j-1})|, & j < i - 1 \\ |C'_i(x_{j+1})| &< \frac{1}{2} |C'_i(x_j)|, & j > i. \end{aligned} \quad (5.10)$$

LEMMA 5.2. Let $\tau(x)$ be any exponential spline with knots $\{x_i\}_{i=1}^{N+1}$ (uniform mesh and tension) which satisfies

$$\tau_{i-1} = \tau_{i+1} = 0, \quad \tau_i = v > 0, \quad \tau'_{i-1} \cdot \tau''_{i-1} \geq 0, \quad \tau'_{i+1} \cdot \tau''_{i+1} \leq 0. \quad (5.11)$$

Then

$$\tau''_i < 0, \quad \tau''_{i-1} \geq 0, \quad \tau'_{i+1} \leq 0; \quad \tau(x) \geq 0, \quad x \in [x_{i-1}, x_{i+1}]. \quad (5.12)$$

LEMMA 5.3. Let $\tau(x)$ be such that

$$\tau_{i-1} = \tau_i = \tau_{i+1} = 0, \quad \tau''_{i-1} \leq 0, \quad \tau''_{i+1} \geq 0. \quad (5.13)$$

Then

$$\tau(x) \geq 0, \quad x \in [x_{i-1}, x_i]. \quad (5.14)$$

COROLLARY 5.4. For $i = 2, \dots, N$, $C_i(x)$ satisfies

$$\begin{aligned} |C_i(x)| &\leq |C'_i(x_j)| \cdot h, & x \in [x_j, x_{j+1}], j > i \\ |C_i(x)| &\leq |C'_i(x_j)| \cdot h, & x \in [x_{j-1}, x_j], j < i - 1. \end{aligned} \quad (5.15)$$

Note that these results imply an exponential decay of $|C_i(x)|$ away from x_i .

We next consider the "natural cardinal splines defined by the conditions

$$N_i(x_j) = \delta_{ij}, \quad N''_i(x_1) = N''_i(x_{N+1}) = 0, \quad (i, j = 1, \dots, N + 1), \quad (5.16)$$

which form a basis for the "natural" splines $\tau(x)$ satisfying $\tau''(x_1) = \tau''(x_{N+1}) = 0$. Let $\bar{N}(x)$ be any exponential spline with $\bar{N}''_1 = \bar{N}''_{N+1} = 0$. The remaining $\{\bar{N}''_i\}_{i=2}^N$ are determined from

$$\begin{bmatrix} a & 1 & \cdots & 0 \\ 1 & a & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & a & 1 \\ 0 & \cdots & & 1 & a \end{bmatrix} \begin{bmatrix} \bar{N}''_2 \\ \bar{N}''_3 \\ \vdots \\ \bar{N}''_{N-1} \\ \bar{N}''_N \end{bmatrix} = \begin{bmatrix} b_2/e \\ b_3/e \\ \vdots \\ b_{N-1}/e \\ b_N/e \end{bmatrix}, \quad (5.17)$$

or $A\bar{N}'' = \bar{b}$, where $a = 2d/e$, which is obtained from (3.5) with modified end conditions and uniformity over the intervals. Now, A can be factored as $A = LU$, where

$$L = \begin{bmatrix} \rho_2 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \rho_N \end{bmatrix}; \quad U = \begin{bmatrix} 1 & \rho_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \rho_{N-1}^{-1} \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \quad (5.18)$$

with $\rho_2 = a$, $\rho_i = a - \rho_{i-1}^{-1}$ ($i = 3, \dots, N$). It follows that

$$\begin{aligned} \frac{2d}{e} = a = \rho_2 > \rho_3 > \cdots > \rho_{N-1} > \rho_N > \frac{a + \sqrt{a^2 - 4}}{2} \\ &= \frac{d}{e} + \sqrt{\left(\frac{d}{e}\right)^2 - 1} \geq 2 + \sqrt{3}. \end{aligned}$$

Thus, $\det A = \prod_i \rho_i > 0$ since each $\rho_i > 0$. Hence, we conclude that A is nonsingular and that these "natural" cardinal splines are well-defined.

In particular, letting $\bar{N}(x) = N_j(x)$ we have (after some lengthy computations) the following recurrence relations:

$$\left. \begin{aligned} \bar{N}''_i &= -\rho_i^{-1} \bar{N}''_{i+1} & i = 2, \dots, j-2 \\ \bar{N}''_i &= -\rho_{N+1-i}^{-1} \bar{N}''_{i-1} & i = j+2, \dots, N \\ \bar{N}''_{j-1} &= \rho_{i-1}^{-1} [(he)^{-1} - \bar{N}''_j] \\ \bar{N}''_{j+1} &= \rho_{N-j}^{-1} [(he)^{-1} - N''_j] \\ \bar{N}''_j &= -(he)^{-1} \left[\frac{2 + \rho_{j-1}^{-1} + \rho_{N-j}^{-1}}{4 - \rho_{j-1}^{-1} - \rho_{N-j}^{-1}} \right] \end{aligned} \right\} \quad (5.19) \quad j \neq 1, 2, N, N+1.$$

We then have the following.

LEMMA 5.4. For $\bar{N}(x) = N_j(x)$ ($j = 3, \dots, N-1$) and either $x \in [x_i, x_{i-1}]$, $j+1 \leq i \leq N$ or $x \in [x_{i-1}, x_i]$, $2 \leq i \leq j-1$, we have

$$|\bar{N}(x)| \leq [1 - 1/\sqrt{3}] \cdot h^2 \cdot |\bar{N}_i''|. \quad (5.20)$$

Proof. We consider the case $x \in [x_{i-1}, x_i]$, $2 \leq i \leq j-1$. Then $\bar{N}_{i-1}'' = -\rho_{i-1}^{-1} \bar{N}_i''$ and, from (3.3),

$$\begin{aligned} \bar{N}(x) &= \frac{N_i''}{p^2} \left\{ \left[\frac{\sinh p(x-x_{i-1})}{\sinh ph} - \rho_{i-1}^{-1} \cdot \frac{\sinh p(x_i-x)}{\sinh ph} \right] \right. \\ &\quad \left. - \left[\frac{x-x_{i-1}}{h} - \rho_{i-1}^{-1} \cdot \frac{x_i-x}{h} \right] \right\} \\ &\Rightarrow |\bar{N}(x)| \leq [1 - 1/\sqrt{3}] \cdot h^2 \cdot |\bar{N}_i''|. \end{aligned} \quad (5.21)$$

Similar considerations apply to the case $x \in [x_i, x_{i+1}]$, $j+1 \leq i \leq N$. ■

COROLLARY 5.5. For the above considered case we have

$$|\bar{N}(x)| \leq \frac{2}{3} \cdot \frac{h}{e} (\sqrt{3}-1)(2-\sqrt{3})^{j-i}, \quad (5.22)$$

with a similar result for the other case.

Proof. The recurrence relations for $\{\bar{N}_k''\}_{k=2}^j \Rightarrow$

$$|\bar{N}_i''| \leq \frac{2}{\sqrt{3}he} \cdot (2-\sqrt{3})^{j-i}. \quad (5.23)$$

Substitution into (5.21) yields (5.22). Thus we also have exponential decay of these "natural" cardinal basis splines. ■

The cardinal spline formulation is not overly useful for calculations because of the global nature of the basis functions. However, it does provide great insight into many numerical aspects of splines. For example, the above considerations allow us to determine the effect of a change in some data value, say $f_i \leftarrow f_i + \varepsilon_i$. We simply add the term $\varepsilon_i \cdot C_i(x)$ to the existing cardinal spline expansion. Similarly, a study of $C_0(x)$ and $C_{N+2}(x)$ allows one to discuss the global effect of end conditions.

Next, we introduce another basis for the space of exponential splines. This basis will be constructed so as to have minimal support [2, 5, 11, 13]. We begin with the following

THEOREM 5.1. *Any exponential spline, $B(x)$, with a support of fewer than four intervals is identically zero (assuming there are at least five knots).*

Proof. Similar to that for cubic splines [7]. ■

We now proceed to construct such a "B-spline" with a support of four intervals. Given $\{x_0 + ih\}_{i=-2}^2$ we require that

$$\begin{aligned} B(x_0 - 2h) = B'(x_0 - 2h) = B''(x_0 - 2h) = B(x_0 + 2h) \\ = B'(x_0 + 2h) = B''(x_0 + 2h) = 0. \end{aligned} \quad (5.24)$$

Also, we normalize so that $B(x_0) = 1$. By symmetry, we set $B'(x_0) = 0$. This allows us to solve for $B(x)$ on $[x_0, x_0 + 2h]$ and then reflect the result about the line $x = x_0$. Applying the end conditions and continuity conditions we arrive at

$$B(x) = \begin{cases} a_1 + b_1(x - x_0) + c_1 e^{p(x - x_0)} + d_1 e^{-p(x - x_0)}, & x_0 \leq x \leq x_0 + h \\ b_2[(x - x_0 - 2h) - (1/p) \sinh p(x - x_0 - 2h)], & x_0 + h \leq x \leq x_0 + 2h \end{cases} \quad (5.25)$$

$$B'(x) = \begin{cases} b_1 + pc_1 e^{p(x - x_0)} - pd_1 e^{-p(x - x_0)}, & x_0 \leq x \leq x_0 + h \\ b_2[1 - \cosh p(x - x_0 - 2h)], & x_0 + h \leq x \leq x_0 + 2h \end{cases} \quad (5.26)$$

$$B''(x) = \begin{cases} p^2 c_1 e^{p(x - x_0)} + p^2 d_1 e^{-p(x - x_0)}, & x_0 \leq x \leq x_0 + h \\ -b_2 \sinh p(x - x_0 - 2h), & x_0 + h \leq x \leq x_0 + 2h, \end{cases} \quad (5.27)$$

where

$$\begin{aligned} b_2 &= \frac{p}{2(phc - s)}, & a_1 &= \frac{phc}{phc - s}, \\ b_1 &= \frac{p}{2} \cdot \left[\frac{c(c - 1) + s^2}{(phc - s)(1 - c)} \right], & c_1 &= \frac{1}{4} \left[\frac{e^{-ph}(1 - c) + s(e^{-ph} - 1)}{(phc - s)(1 - c)} \right], \\ d_1 &= \frac{1}{4} \left[\frac{e^{ph}(c - 1) + s(e^{ph} - 1)}{(phc - s)(1 - c)} \right]. \end{aligned} \quad (5.28)$$

Next add the points $x_{-2}, x_{-1}, x_0, x_{N+2}, x_{N+3}, x_{N+4}$ to the set of knots in the obvious fashion. Denote by $B_i(x)$ the B-spline centered at x_i ($i = 0, \dots, N + 2$). All the B_i so defined are simply translates of this canonical B-spline. A straightforward computation confirms that indeed $\{B_i(x)\}_{i=0}^{N+2}$ forms a basis on this mesh with tension p .

Let $\tau(x) = \sum_{i=0}^{N+2} a_i B_i(x)$. The linear systems which follow represent interpolation at the nodes of the B-splines defined above.

Consider the case of slope end conditions. Using (5.25) and (5.26) to interpolate to $[f'_1, f_1, \dots, f_{N+1}, f'_{N+1}]$ yields

$$\begin{bmatrix} \frac{p(1-c)}{2(phc-s)} & & & & & \\ & 0 & & & & \\ \frac{s-ph}{2(phc-s)} & & & & & \\ & & & & & \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & \dots & \frac{s-ph}{2(phc-s)} & & 1 & \frac{s-ph}{2(phc-s)} \\ & & & & & \\ 0 & \dots & \frac{p(1-c)}{2(phc-s)} & & 0 & \frac{p(c-1)}{2(phc-s)} \end{bmatrix}$$

$$\times \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \\ a_{N-2} \end{bmatrix} = \begin{bmatrix} f'_1 \\ f_1 \\ \vdots \\ f_{N+1} \\ f'_{N+1} \end{bmatrix}. \quad (5.29)$$

This system is irreducibly diagonally dominant and hence nonsingular so that $\tau(x)$ is uniquely expressible in this form. Note that we can reduce this system to tridiagonal form by eliminating a_0 and a_{N+2} . Simply multiply the first equation by $(s-ph)/p(c-1)$ and add it to the second equation and multiply the last equation by $(s-ph)/p(1-c)$ and add it to the equation above it.

Consider the case of second derivative end conditions. Using (5.25) and (5.27) to interpolate to $[f''_1, f_1, \dots, f_{N-1}, f''_{N+1}]$ yields

$$\begin{bmatrix} \frac{p^2s}{2(phc-s)} & & & & & \\ & \frac{-p^2s}{phc-s} & & & & \\ \frac{s-ph}{2(phc-s)} & & & & & \\ & & & & & \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & \dots & \frac{s-ph}{2(phc-s)} & & 1 & \frac{s-ph}{2(phc-s)} \\ & & & & & \\ 0 & \dots & \frac{p^2s}{2(phc-s)} & & \frac{-p^2s}{phc-s} & \frac{p^2s}{2(phc-s)} \end{bmatrix}$$

$$\times \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N+1} \\ a_{N+2} \end{bmatrix} = \begin{bmatrix} f''_1 \\ f_1 \\ \vdots \\ f_{N+1} \\ f''_{N+1} \end{bmatrix}. \quad (5.30)$$

Note that we can also reduce this system to tridiagonal form by eliminating a_0 and a_{N+2} . Simply multiply the first equation by $(ph-s)/p^2s$ and add it to the second equation and multiply the last equation by $(ph-s)/p^2s$ and add it to the equation above it. The resulting system is strictly diagonally dominant and hence nonsingular so that $\tau(x)$ is once again seen to be uniquely expressible in this form.

Note that no more than four basis functions contribute to the value of $\tau(x)$ at any point.

In addition to the value of the B-spline representation as a theoretical tool and a computational device, it is also of considerable utility in computer aided design. This utility stems from the local nature of this basis. A spline created as a linear combination of B-splines can be displayed with the user then being able to alter the coefficients of the expansion. As each change affects only four intervals, the user can experiment and design a pleasing curve.

There is an even more subtle aspect of such interactive design. Recall that, from (5.25) and (5.28),

$$\tau_j = \frac{s-ph}{2(phc-s)} a_{j-1} + a_j + \frac{s-ph}{2(phc-s)} a_{j+1}; \quad j = k-2, k, k+1. \quad (5.31)$$

Thus if we consider the local perturbation

$$\tau_k \leftarrow \tau_k + \delta, \tau_{k-1} \leftarrow \tau_{k-1} + \frac{s-ph}{2(phc-s)} \delta, \quad \tau_{k+1} \leftarrow \tau_{k+1} + \frac{s-ph}{2(phc-s)} \delta \quad (5.32)$$

then this simply amounts to redefining $a_k \leftarrow a_k + \delta$. Hence, groups of points may be moved without requiring the solution of the spline matrix equation.

6. PIECEWISE-GENERALIZED SPLINES

Certain important properties of exponential splines are most readily obtained by appealing to a more general framework. Consequently, we next introduce the concept of piecewise-generalized splines thereby arriving at certain extremal properties of the exponential spline.

The symmetric factorization of the exponential spline operator as

$$D^4 - p_i^2 D^2 = (D^2 + p_i D)(D^2 - p_i D) \quad (6.1)$$

permits an interpretation in a generalized spline context [1, 6, 10, 11, 16, 17, 21]. In general, let L be a linear differential operator of order m

$$L = p_0(x) D^m + p_1(x) D^{m-1} + \dots + p_{m-1}(x) D + p_m(x) I \quad (6.2)$$

with $p_k(x) \in C^m[a, b]$ ($k = 0, \dots, m$). Let L^* be the formal adjoint of L :

$$L^* = (-1)^m D^m(p_0) + (-1)^{m-1} D^{m-1}(p_1) + \dots - D(p_{m-1}) + p_m I. \quad (6.3)$$

An interpolatory generalized spline, s , associated with L is defined by the conditions $s \in C^{2m-2}[a, b]$; $s \in K_{2m}(x_i, x_{i+1})$, where it satisfies $L^* L s = 0$ ($i = 1, \dots, N$), and $s(x_i) = f_i$ ($i = 1, \dots, N + 1$). We employ the notation K_{2m} to denote the class of functions with absolutely continuous $(2m - 1)$ th derivatives and a square integrable $(2m)$ th derivatives. Note that generalized splines as defined are a special case of L -splines [16].

We may instead insist that s satisfy $L_i^* L_i s = 0$ on (x_i, x_{i+1}) . In such a case, s is called a piecewise-generalized spline which is a special case of piecewise L -splines [10]. If we let $L_i = D^2 - p_i D$ then we recover the exponential spline.

The principal advantage of this framework is that the functional $\sum_{i=1}^N \int_{x_i}^{x_{i+1}} (L_i f)^2 dx$ is minimized by the piecewise-generalized spline fulfilling the end conditions $(L_1 \tau)^{(k-1)} = 0$ ($k = 1, \dots, m - 1$) at x_1 and $(L_N \tau)^{(k-1)} = 0$ ($k = 1, \dots, m - 1$) at x_{N+1} .

Results in the literature on piecewise L -splines [10] provide certain extremal properties associated with either of the functionals $\int_a^b (f'' \pm p f')^2 dx$. We next derive a third functional, $\int_a^b [(f'')^2 + p^2 (f')^2] dx$, and establish its associated extremal properties.

We begin with the following

LEMMA 6.1.

$$\int_a^b \{v(u'' \pm pu') - u(v'' \mp pv')\} dx = \{u'v - uv' \pm puw\} \Big|_a^b. \quad (6.4)$$

Proof. Integrate by parts twice. ■

Next, let $f \in K_2[a, b]$ and $p(x)$ be a step function with $p(x) = p_i$ on (x_i, x_{i+1}) ($i = 1, \dots, N$). We then have the identity

$$\begin{aligned} & \int_a^b (f'' \pm p f')^2 dx \\ &= \int_a^b [f'' - \tau'' \pm p(f' - \tau')]^2 dx + \int_a^b (\tau'' \pm p \tau')^2 dx \\ &+ 2 \int_a^b [f'' - \tau'' \pm p(f' - \tau')] \cdot (\tau'' \pm p \tau') ds. \end{aligned} \quad (6.5)$$

By letting $u = f - \tau$ and $v = \tau'' \pm p\tau'$ in Lemma 6.1, we arrive at

$$\begin{aligned}
 & \sum_{i=1}^N \int_{x_i}^{x_{i+1}} \{(\tau'' \pm p\tau')[f'' - \tau'' \pm p(f' - \tau')] \\
 & \quad - (f - \tau)[\tau^{(iv)} \pm p\tau'' \mp p(\tau''' \pm p\tau'')]\} dx \\
 & = \sum_{i=1}^N \{ (f' - \tau')(\tau'' \pm p\tau') - (f - \tau) \\
 & \quad \times (\tau''' \pm p\tau'') \pm p(f - \tau)(\tau'' \pm p\tau') \}_{x_i}^{x_{i+1}}. \tag{6.6}
 \end{aligned}$$

If $\tau(x)$ interpolates to $f(x)$ at $\{x_i\}_{i=1}^{N+1}$ then (6.6) reduces to

$$\begin{aligned}
 & \sum_{i=1}^N \int_{x_i}^{x_{i+1}} (\tau'' \pm p\tau')[f'' - \tau'' \pm p(f' - \tau')] dx \\
 & = \sum_{i=1}^N \{ (f' - \tau')(\tau'' \pm p\tau') \}_{x_i}^{x_{i+1}} \\
 & = [f'(b) - \tau'(b)] \cdot [\tau''(b) \pm p\tau'(b)] \\
 & \quad - [f'(a) - \tau'(a)] \cdot [\tau''(a) \pm p\tau'(a)]. \tag{6.7}
 \end{aligned}$$

Applying this to the last term of (6.5), we obtain

$$\begin{aligned}
 & \int_a^b (f'' \pm pf')^2 dx \\
 & = \int_a^b [f'' - \tau'' \pm p(f' - \tau')]^2 dx + \int_a^b (\tau'' \pm p\tau')^2 dx \\
 & \quad + 2\{ [f'(b) - \tau'(b)] \cdot [\tau''(b) \pm p\tau'(b)] \\
 & \quad - [f'(a) - \tau'(a)] \cdot [\tau''(a) \pm p\tau'(a)] \}. \tag{6.8}
 \end{aligned}$$

Adding together the two relations embodied in (6.8) leads to

THEOREM 6.1 (Extended Holladay's Theorem).

$$\begin{aligned}
 & \int_a^b [(f'')^2 + p^2(f')^2] dx \\
 & = \int_a^b [(f'' - \tau'')^2 + p^2(f' - \tau')^2] dx + \int_a^b [(\tau'')^2 + p^2(\tau')^2] dx \\
 & \quad + 2\{ [f'(b) - \tau'(b)] \cdot \tau''(b) - [f'(a) - \tau'(a)] \cdot \tau''(a) \}. \tag{6.9}
 \end{aligned}$$

This leads us to define the inner product [9]

$$\langle f, g \rangle_e = \int_a^b f''(x) g''(x) + p^2(x) f'(x) g'(x) dx \tag{6.10}$$

together with the induced pseudonorm

$$\|f\|_e = \left\{ \int_a^b [(f''(x))^2 + p^2(x)(f'(x))^2] dx \right\}^{1/2}. \tag{6.11}$$

The extended Holladay's theorem (6.9) may then be restated as

$$\begin{aligned} \|f\|_e^2 = & \|f - \tau\|_e^2 + \|\tau\|_e^2 + 2\{ [f'(b) - \tau'(b)] \\ & \cdot \tau''(b) - [f'(a) - \tau'(a)] \cdot \tau''(a) \}. \end{aligned} \tag{6.12}$$

If either $\tau''(a) = \tau''(b) = 0$, or $\tau'(a) = f'(a)$, $\tau'(b) = f'(b)$ then we conclude that $\|f\|_e^2 = \|f - \tau\|_e^2 + \|\tau\|_e^2$ thus implying the important

THEOREM 6.2 (Minimum Norm Property). *Given $f(x) \in K_2[a, b]$, of all $g(x) \in K_2[a, b]$ satisfying either $g''(a) = g''(b) = 0$ or $g'(a) = f'(a)$, $g'(b) = f'(b)$ and interpolating to $f(x)$ at $\{x_i\}_{i=1}^{n+1}$, the one with minimum pseudonorm is the interpolatory exponential spline, $\tau(x)$.*

Moreover, we obtain the equally important

THEOREM 6.3 (Best Approximation Property). *Let $f(x) \in K_2[a, b]$, $\tau(x)$ be the interpolatory exponential spline satisfying $\tau'(a) = f'(a)$, $\tau'(b) = f'(b)$, and $\bar{\tau}(x)$ be any other exponential spline with the same mesh and tension. Then, $\|f - \bar{\tau}\|_e \geq \|f - \tau\|_e$. That is to say, $\tau(x)$ is the best approximation by exponential splines.*

Proof. Let $g = f - \bar{\tau}$ and $\hat{\tau} = \tau - \bar{\tau}$. Hence, $\hat{\tau}$ is an exponential spline satisfying $\hat{\tau}(x_i) = g(x_i)$, $\hat{\tau}'(a) = g'(a)$, and $\hat{\tau}'(b) = g'(b)$. By Theorem 6.1, we have

$$\|g\|_e^2 = \|g - \hat{\tau}\|_e^2 + \|\hat{\tau}\|_e^2 \Rightarrow \|f - \hat{\tau}\|_e^2 = \|f - \tau\|_e^2 + \|\tau - \bar{\tau}\|_e^2. \quad \blacksquare \tag{6.13}$$

7. EXPONENTIAL HERMITE INTERPOLANTS

Spline interpolation by its very nature is a global scheme as it entails the solution of a tridiagonal system. On the other hand, osculatory (Hermite) interpolation provides a local means of interpolation. For this reason, Hermite interpolation is many times preferred over spline interpolation.

Hermite interpolation requires the specification of a certain number of consecutive derivatives at each knot. The particular number may vary from

knot to knot. The local nature of this approximation comes to us at the expense of smoothness. For example, if in the polynomial case we also specify first derivatives we have a cubic Hermite interpolant which is only C^1 as opposed to the C^2 smoothness provided by the cubic spline. Moreover, the required derivatives are typically not available and must themselves be approximated.

With these provisos duly noted, we now proceed to discuss Hermite interpolation by piecewise exponentials. For ease of presentation, we restrict $x \in [0, 1]$ with $f(0) = f_0$, $f(1) = f_1$, $f'(0) = f'_0$, $f'(1) = f'_1$ given. In this setting, exponential Hermite interpolation is effected by

$$h(x) = f_0\phi_0(x) + f_1\phi_1(x) + f'_0\bar{\phi}_0(x) + f'_1\bar{\phi}_1(x), \quad (7.1)$$

where ϕ_0 , ϕ_1 , $\bar{\phi}_0$, $\bar{\phi}_1$ are the cardinal functions defined by

$$\begin{aligned} \phi_0(x) &= a_0 + b_0x + c_0e^{px} + d_0e^{-px}; & \phi_0(0) &= 1, \phi_0(1) = \phi'_0(0) = \phi'_0(1) = 0 \\ \phi_1(x) &= a_1 + b_1x + c_1e^{px} + d_1e^{-px}; & \phi_1(1) &= 1, \phi_1(0) = \phi'_1(0) = \phi'_1(1) = 0 \\ \bar{\phi}_0(x) &= \bar{a}_0 + \bar{b}_0x + \bar{c}_0e^{px} + \bar{d}_0e^{-px}; & \bar{\phi}'_0(0) &= 1, \bar{\phi}_0(0) = \bar{\phi}_0(1) = \bar{\phi}'_0(1) = 0 \\ \bar{\phi}_1(x) &= \bar{a}_1 + \bar{b}_1x + \bar{c}_1e^{px} + \bar{d}_1e^{-px}; & \bar{\phi}'_1(1) &= 1, \bar{\phi}_1(0) = \bar{\phi}_1(1) = \bar{\phi}'_1(0) = 0. \end{aligned} \quad (7.2)$$

This leads to

$$\begin{aligned} \phi_0(x) &= \frac{1}{\mathcal{D}} \{2(1 - \cosh p + p \sinh p) + (-2p \sinh p)x \\ &\quad + (1 - e^{-p})e^{px} + (1 - e^p)e^{-px}\} \\ \phi_1(x) &= \frac{1}{D} \{2(1 - \cosh p) + (2p \sinh p)x \\ &\quad - (1 - e^{-p})e^{px} - (1 - e^p)e^{-px}\} \\ \bar{\phi}_0(x) &= \frac{-1}{p\mathcal{D}} \{2(\sinh p - p \cosh p) - 2p(1 - \cosh p)x \\ &\quad - (1 - e^{-p} - pe^{-p})e^{px} + (1 - e^p + pe^p)e^{-px}\} \\ \bar{\phi}_1(x) &= \frac{1}{p\mathcal{D}} \{2(\sinh p - p) + 2p(1 - \cosh p)x \\ &\quad + (-1 + p + e^{-p})e^{px} + (1 + p - e^p)e^{-px}\}, \end{aligned} \quad (7.3)$$

where $\mathcal{D} = 4 - 4 \cosh p + 2p \sinh p$. Note that

$$\phi_1(x) = \phi_0(1 - x), \quad \bar{\phi}_1(x) = -\bar{\phi}_0(1 - x) \quad (7.4)$$

and that all four basis functions involve e^{px} and e^{-px} . The computational complexity may be reduced by the following construction.

Let

$$h(x) = f_0\psi_1(x) + f_1\tilde{\psi}_1(x) + \mathcal{A}\psi_2(x) + \mathcal{B}\psi_3(x), \quad (7.5)$$

where

$$\begin{aligned} \psi_1(x) &= 1 - x \\ \tilde{\psi}_1(x) &= x = \psi_1(1 - x) \\ \psi_2(x) &= a_2 + b_2x + c_2e^{px}; & \psi_2(0) = 0, \psi_2(1) = 0, \psi_2'(0) = \alpha \\ \psi_3(x) &= a_3 + b_3x + c_3e^{-px}; & \psi_3(0) = 0, \psi_3(1) = 0, \psi_3'(0) = \beta \end{aligned} \quad (7.6)$$

and \mathcal{A} , \mathcal{B} are as yet undetermined.

This produces

$$\begin{aligned} \psi_2(x) &= \alpha \cdot \frac{-1 + (1 - e^p)x + e^{px}}{1 + p - e^p} \\ \psi_3(x) &= \beta \cdot \frac{-1 + (1 - e^{-p})x + e^{-px}}{1 - p - e^{-p}}. \end{aligned} \quad (7.7)$$

Letting $q = (1 - e^p + pe^p)$, $(1 + p - e^p)$ we have

$$\psi_2'(1) = q \cdot \psi_2'(0), \quad \psi_3'(0) = q \cdot \psi_3'(1). \quad (7.8)$$

Now, $f_0\psi_1 + f_1\tilde{\psi}_1$ has a slope of $m = f_1 - f_0$. Hence, we require that

$$\mathcal{A}\psi_2'(0) + \mathcal{B}\psi_3'(0) = f_0' - m, \quad \mathcal{A}\psi_2'(1) + \mathcal{B}\psi_3'(1) = f_1' - m \Rightarrow \quad (7.9)$$

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-q} & \frac{-q}{1-q} \\ \frac{-q}{1+q} & \frac{1}{1+q} \end{bmatrix} \begin{bmatrix} f_0' - m \\ f_1' - m \end{bmatrix}. \quad (7.10)$$

Equally as simple would be to let $\psi_2'(0) = \psi_3'(1) = 1$, in which case

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-q^2} & \frac{-q}{1-q^2} \\ \frac{-q}{1-q^2} & \frac{1}{1-q^2} \end{bmatrix} \begin{bmatrix} f_0' - m \\ f_1' - m \end{bmatrix}. \quad (7.11)$$

Equally important as the result is the technique. Specifically, suppose that we are provided with two new functions $e^{\sigma x}$ and $e^{-\sigma x}$. Together with

1, x , $e^{\rho x}$, $e^{-\rho x}$, we are now required to match f, f', f'' at $x=0, 1$. The cardinal spline approach would entail constructing the six new basis functions $\phi_0, \phi_1, \bar{\phi}_0, \bar{\phi}_1, \hat{\phi}_0, \hat{\phi}_1$ defined by

$$\begin{aligned}
\phi_0(x) &= a_0 + b_0 x + c_0 e^{\rho x} + d_0 e^{-\rho x} + e_0 e^{\sigma x} + f_0 e^{-\sigma x}; \\
\phi_0(0) &= 1, \phi_0(1) = \phi_0'(0) = \phi_0'(1) = \phi_0''(0) = \phi_0''(1) = 0 \\
\phi_1(x) &= \phi_0(1-x) \\
\bar{\phi}_0(x) &= \bar{a}_0 + \bar{b}_0 x + \bar{c}_0 e^{\rho x} + \bar{d}_0 e^{-\rho x} + \bar{e}_0 e^{\sigma x} + \bar{f}_0 e^{-\sigma x}; \\
\bar{\phi}_0(0) &= 1, \bar{\phi}_0(1) = \bar{\phi}_0'(1) = \bar{\phi}_0''(0) = \bar{\phi}_0''(1) = 0 \quad (7.12) \\
\bar{\phi}_1(x) &= -\bar{\phi}_0(1-x) \\
\hat{\phi}_0(x) &= \hat{a}_0 + \hat{b}_0 x + \hat{c}_0 e^{\rho x} + \hat{d}_0 e^{-\rho x} + \hat{e}_0 e^{\sigma x} + \hat{f}_0 e^{-\sigma x}; \\
\hat{\phi}_0''(0) &= 1, \hat{\phi}_0(0) = \hat{\phi}_0(1) = \hat{\phi}_0'(1) = \hat{\phi}_0'(0) = \hat{\phi}_0''(1) = 0 \\
\hat{\phi}_1(x) &= \hat{\phi}_0(1-x).
\end{aligned}$$

On the other hand, our alternative technique involves calculating $h''(0)$ and $h''(1)$ and subsequently defining

$$\begin{aligned}
\psi_4(x) &= a_4 + b_4 x + c_4 e^{\rho x} + d_4 e^{-\rho x} + e_4 e^{\sigma x}; \\
\psi_4(0) &= \psi_4(1) = \psi_4'(0) = \psi_4'(1) = 0, \psi_4''(0) = \alpha \quad (7.13) \\
\psi_5(x) &= a_5 + b_5 x + c_5 e^{\rho x} + d_5 e^{-\rho x} + e_5 e^{-\sigma x}; \\
\psi_5(0) &= \psi_5(1) = \psi_5'(0) = \psi_5'(1) = 0, \psi_5''(0) = \beta.
\end{aligned}$$

Finally, we would set

$$H(x) = h(x) + \mathcal{A}\psi_4(x) + \mathcal{B}\psi_5(x) \quad (7.14)$$

where

$$\mathcal{A}\psi_4''(0) + \mathcal{B}\psi_5''(0) = f_0'' - h''(0), \quad \mathcal{A}\psi_4''(1) + \mathcal{B}\psi_5''(1) = f_1'' - h''(1). \quad (7.15)$$

8. HIGHER ORDER TENSION INTERPOLANTS

In this section, we generalize the exponential spline and Hermite interpolant previously described. The starting point for this discussion is the characterization of these exponential interpolants as belonging to the null space of $E = D^4 - p_i^2 D^2$ between knots.

One possible extension would be to consider piecewise solutions of

$$[D^4 + \alpha \cdot p_i D^3 - p_i^2 D^2] t(x) = 0. \quad (8.1)$$

However, this operator does not permit a factorization as L^*L since $\alpha \neq 0$ produces an operator that is not selfadjoint. As such, it does not generate piecewise-generalized splines. In fact, a direct computation establishes that the most general fourth order homogeneous differential operator with real constant coefficients (lead coefficient = 1) that permits such a decomposition is precisely the exponential spline operator, E .

Hence, if we want a generalization using constant coefficients that produces piecewise-generalized splines we must increase the order of the differential operator. Thus, consider the sixth order operator equation

$$[D^6 + \alpha D^5 + \beta D^4 + \gamma D^3 + \delta D^2] t(x) = 0. \quad (8.2)$$

Let $L = D^3 + \mu D^2 + \eta D \Rightarrow L^* = -D^3 + \mu D^2 - \eta D$. Thus, $L^*L = -D^6 + (\mu^2 - 2\eta) D^4 - \eta^2 D^2$. We thus arrive at

$$[L^*L] t(x) = 0, \quad (8.3)$$

where $\alpha = \gamma = 0$ (by selfadjointness of the differential operator), $\delta = \eta^2$, $\beta = 2\eta - \mu^2$. That is,

$$[D^6 + (2\eta - \mu^2) D^4 + \eta^2 D^2] t(x) = 0 \quad (8.4)$$

or, in factored form,

$$[D^2(D^2 + \mu D + \eta I)(D^2 - \mu D + \eta I)] t(x) = 0. \quad (8.5)$$

The characteristic roots of this operator will then determine the basis functions for the null space. The double root of zero admits 1 and x . The other roots are

$$\lambda \in \left\{ \frac{-\mu + \sqrt{\mu^2 - 4\eta}}{2}, \frac{\mu \pm \sqrt{\mu^2 - 4\eta}}{2} \right\}. \quad (8.6)$$

If $\eta = 0$, we admit x^2 , x^3 , $e^{-\mu x}$. If $\mu^2 - 4\eta = 0$, we admit $e^{-\mu x/2}$, $x e^{-\mu x/2}$, $e^{\mu x/2}$, $x e^{\mu x/2}$. Otherwise, we have four distinct λ 's and corresponding basis functions. Note that for $\mu^2 > 4\eta$ we obtain hyperbolic functions while if $\mu^2 < 4\eta$ we obtain trigonometric functions.

The above considerations lead us to the following definition of tension interpolants of order $2m$ (degree $2m - 1$). Let

$$T = D^{2m} + \alpha_{2m-2} D^{2m-2} + \dots + \alpha_4 D^4 + \alpha_2 D^2 \quad (8.7)$$

possess the factorization $T = (-1)^m L^*L$, where

$$L = D^m + \beta_{m-1} D^{m-1} + \dots + \beta_2 D^2 + \beta_1 D \Rightarrow \quad (8.8)$$

$$L^* = (-1)^m D^m + (-1)^{m-1} \beta_{m-1} D^{m-1} + \dots + \beta_2 D^2 - \beta_1 D. \quad (8.9)$$

We then say that $t(x)$ is a tension interpolant if it is a piecewise solution of

$$[T] t(x) = [(-1)^m L^* L] t(x) = 0. \quad (8.10)$$

In this case

$$\begin{aligned} & D^2[D^{m-1} + \beta_{m-1}D^{m-2} + \dots + \beta_2D + \beta_1I]. \\ [D^{m-1} - \beta_{m-1}D^{m-2} + \dots + (-1)^{m-2}\beta_2D + (-1)^{m-1}\beta_1I] t(x) &= 0. \end{aligned} \quad (8.11)$$

Again, the double root of zero admits 1 and x as basis functions for $N(L^*L)$. The other basis functions are determined by the remaining roots of the characteristic equation.

This generalization allows us to pursue one of two routes. First, we could require a greater degree of smoothness at the knots. In this context, the sixth order operator discussed above would produce a quintic tension spline which is $C^4[a, b]$ ($C^{2m-2}[a, b]$ in the general case). Secondly, we could require higher order interpolation at the knots. Our previous example then amounts to a quintic Hermite interpolant under tension matching function values together with first and second derivatives which is $C^2[a, b]$ ($C^{m-1}[a, b]$ in the general case).

9. CONCLUSION

In the preceding paragraphs, we have accomplished two objectives. First, building on the previous work of Pruess, we have further extended the theory of exponential splines. Most notable in this regard are the treatments of convergence of third derivatives, cardinal and B-spline bases, and extremal properties. Second, the exponential spline has been generalized in two directions. Higher order tension splines have been defined thus providing the generalization of quintic splines, etc. Higher degree interpolation has been treated thus providing the generalization of Hermite polynomial interpolation.

As previously noted, these investigations were motivated by Pruess' results on co-convex and co-monotone interpolation by exponential splines. In addition to the theoretical results presented here, this has also led to the construction of practical tension parameter selection algorithms as well as an in-depth study of computational issues in the use of exponential splines [8]. Furthermore, the application of exponential splines to a broad spectrum of problems in computational fluid dynamics has been pursued [7].

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